NOTES ON F-SINGULARITIES

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Contents

1. Introduction

The action of Frobenius on a ring of positive characteristic has a long history of being used to characterize the singularities of the associated varieties. Work of Kunz shows that a Noetherian ring R of characteristic $p > 0$ is regular if and only if the Frobenius map on R is flat [Kun76]. The use of Frobenius was also applied to several important questions, for example the study of cohomological dimension [HS77] and the study of invariants rings under group actions [HR74] in positive characteristic.

With the development of tight closure theory [HH90] [HH94a] [HH94b] there was an explosion in the understanding of singularities via the Frobenius map, and a number of classes of singularities were formally introduced, which include F-regular, F-rational, F-pure and F-injective singularities. In this note we give an introduction on these "F-singularities", with a focus on the connection with Frobenius actions on local cohomology modules.

Throughout this note, unless otherwise stated, all rings are assumed to be commutative, Noetherian, with multiplicative identity, and of prime characteristic $p > 0$.

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2. Frobenius and F-finite rings

Rings of prime characteristic $p > 0$ come equipped with a special endomorphism, namely the Frobenius endomorphism $F : R \to R$ defined by $F(r) = r^p$. For each $e \in \mathbb{N}$ we can iterate the Frobenius endomorphism e times and obtain the eth Frobenius endomorphism $F^e : R \to R$ defined by $F^e(r) = r^{p^e}$ for each $e \in \mathbb{N}$. Roughly speaking, the study of prime characteristic rings is often the study of algebraic and geometric properties of the Frobenius endomorphism. For example, it is easy to see that R is reduced if and only if $F : R \to R$ is injective.

Suppose that R is reduced and let K be the total ring of fractions of R, thus $K = \prod K_i$ is a product of fields. Let $\overline{K} = \prod \overline{K_i}$. There are inclusions $R \subseteq K \subseteq \overline{K}$. We let

$$R^{1/p^e} = \{s \in \overline{K} \mid s^{p^e} \in R\}.$$

In other words, R^{1/p^e} is the collection of p^eth roots of elements of R. Then R^{1/p^e} is unique up to non-unique isomorphism, and $\varphi_e : R \to R^{1/p^e}$ defined by $\varphi_e(r) = r^{1/p^e}$ is an isomorphism of rings. We can view the Frobenius map as the natural inclusion $R \subseteq R^{1/p^e}$.

Definition 2.1. R is called *F-finite* if for some (or equivalently, every) $e > 0$, the Frobenius map F^e: $R \to R$ is a finite morphism, i.e., the target R is a finitely generated as a module over the source R.

It turns out the F-finiteness can be checked by passing to the reduced ring:

Exercise 1. Prove that R is F-finite if and only if $R/\sqrt{0}$ is F-finite.

The next set of exercises are standard, but they show that F-finite rings are ubiquitous.

Exercise 2. Let R be an F-finite ring. Prove the following:

(1) If $I \subseteq R$ an ideal then R/I is F-finite.
(2) If W a multiplicative subset of R then $W^{-1}R$ is F-finite.
(3) If x an indeterminate then $R[x]$ and $R[[x]]$ are F-finite.

As a consequence, we have

(1) If R is essentially of finite type over an F-finite field (e.g., a perfect field), then R is F-finite.
(2) If $(R, \mathfrak{m})$ is a complete local ring, then R is F-finite if and only if $R/\mathfrak{m}$ is F-finite.

Exercise 3. Let

$$S = \mathbb{F}_p[x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3}, x_{4,1} \ldots],$$

and let W be the multiplicative set

$$W = S - \Big((x_{1,1}) \cup (x_{2,1}, x_{2,2}) \cup (x_{3,1}, x_{3,2}, x_{3,3}) \cup \cdots\Big).$$

Nagata has shown that $R = W^{-1}S$ is a Noetherian domain of infinite Krull dimension [Nag62]. Show that R is not F-finite.

Recall that R is said to be *excellent* if R satisfies the following:

(1) R is universally catenary.
(2) For each $P \in \operatorname{Spec}(R)$ the map $R_P \to \widehat{R_P}$ has geometrically regular fibers.
(3) If S is an R-algebra of finite type then the regular locus of S is an open subset of $\operatorname{Spec}(S)$.

The F-finite property implies the rings are reasonably good:

Theorem 2.2 ([Kun76])**.** *If R is F-finite, then R is excellent. Moreover, if $(R, \mathfrak{m})$ is local, then R is F-finite if and only if R is excellent and $R/\mathfrak{m}$ is F-finite.*

Theorem 2.3 ([Gab04])**.** *If R is F-finite, then R is a homomorphic image of a regular ring.*

Throughout the rest of this note, we will mainly work with F-finite rings. As mentioned in the introduction, the singularities of R are often studied via behavior of the Frobenius endomorphism. A fundamental result in this direction is Kunz's theorem:

Theorem 2.4 ([Kun69])**.** *R is regular if and only if the Frobenius map F^e:$R \to R$ is flat for some (or equivalently, for all) $e > 0$.*

Below we prove the easier direction.

Proof. Assuming R is regular, we want to show that R^{1/p^e} is a flat R-module. We may assume R is local and complete (because flatness can be checked locally and by passing to completion). By Cohen's structure theorem, $R \cong k[[x_1, \ldots, x_d]]$. Now if k is perfect, then R^{1/p^e} is a free R-module with $\{x_1^{i_1/p^e} \cdots x_d^{i_d/p^e} \mid 0 \le i_j < p^e\}$ a free basis. We leave the general case (i.e., k not necessarily perfect) as an exercise. $\square$

Exercise 4. Find an example of a non-local F-finite regular R such that R^{1/p^e} is not free as an R-module for some $e \in \mathbb{N}$.

3. F-pure rings and Fedder's criterion

Definition 3.1. R is called *F-pure* if R is reduced and the natural map $N \to N \otimes_R R^{1/p^e}$ is injective for all R-modules N (for some, or equivalently, all $e > 0$). R is called *F-split* if $R \to R^{1/p^e}$ splits as a map of R-modules (for some, or equivalently, all $e > 0$).

It is easy to see that F-split implies F-pure, the converse holds in many cases.

Exercise 5. Show that an F-pure ring R is F-split if either of the following conditions is satisfied:

(1) R is F-finite.
(2) $(R, \mathfrak{m})$ is complete local.

Despite the above, there are examples of F-pure rings that are not F-split.

Theorem 3.2 ([DS16])**.** *Let V be a DVR whose fraction field is F-finite. Then the following are equivalent:*

(1) V is F-split.
(2) V is F-finite.
(3) V is excellent.

By Kunz's theorem, regular rings are always F-pure ($R \to R^{1/p^e}$ is faithfully flat if R is regular). The above result says that any non-excellent DVR (with F-finite fraction field) is F-pure but not F-split. These examples do exist and we refer to [DS16] for more details.

Open Problem 1. *Does there exist excellent local F-pure rings that are not F-split?*

We next state and sketch the proof of a fundamental result of Fedder.

Theorem 3.3 (Fedder's criterion, [Fed83])**.** *Let $(S, \mathfrak{m})$ be a regular local ring (resp. a standard graded polynomial ring over a field) and let $I \subseteq S$ be an ideal (resp. a homogeneous ideal). Then S/I is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ where $I^{[p]}$ is the ideal generated by p-th powers of elements of I.*

Outline of proof when S is F-finite. There are 4 steps.

(1) We have $\mathrm{Hom}_S(S^{1/p}, S) \cong S^{1/p}$. Let Φ be the generator of $\mathrm{Hom}_S(S^{1/p}, S)$. We detailed this step in the next exercise.
(2) By Kunz's theorem, $S^{1/p}$ is a finite free S-module, every map $(S/I)^{1/p} \to S/I$ can be lifted to a map $S^{1/p} \to S$, thus can be written as $\Phi(s^{1/p} \cdot -)$ for some $s^{1/p} \in S^{1/p}$.
(3) We have $\Phi(s^{1/p} \cdot -)$ induces a map $(S/I)^{1/p} \to S/I$ if and only if $s \in (I^{[p]} : I)$.
(4) We have $\Phi(s^{1/p} \cdot -)$ is surjective if and only if $s \notin \mathfrak{m}^{[p]}$.

Therefore S/I is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. $\square$

Exercise 6. Let S be either the regular local ring $k[[x_1, \dots, x_d]]$ or the standard graded polynomial ring $k[x_1, \dots, x_d]$ where k is a perfect field. We know that S^{1/p^e} is a free S-module with basis $\{x_1^{i_1/p^e} \cdots x_d^{i_d/p^e} \mid 0 \le i_j < p^e\}$.

(1) Show that for each tuple $(i_1, \dots, i_d)$ with $0 \le i_j < p^e$ there is a Frobenius splitting $\varphi_{(i_1,\dots,i_d)} : S^{1/p^e} \to S$ which is the S-linear map defined on basis elements as follows:

$$\varphi_{(i_1,\dots,i_d)}(x_1^{j_1/p^e} \cdots x_d^{j_d/p^e}) = \begin{cases} 1 & (j_1, \dots, j_d) = (i_1, \dots, i_d) \\ 0 & (j_1, \dots, j_d) \neq (i_1, \dots, i_d) \end{cases}.$$

(2) Show that $\operatorname{Hom}_S(S^{1/p^e}, S) \cong S^{1/p^e} \cdot \Phi_e$ where $\Phi_e = \varphi_{(p^e-1,\dots,p^e-1)}$.

Fedder's criterion is extremely useful. We give some examples of F-pure rings.

Example 3.4. (1) Stanley-Reisner rings are F-pure (i.e., polynomial rings mod square free monomial ideal). The key point is that, since $x_1x_2\cdots x_d$ is a multiple of every square free monomial, $(x_1\cdots x_d)^{p-1} \cdot f \in (f^p)$ for any square free monomial f. Thus $(x_1\cdots x_d)^{p-1} \in (I^{[p]} : I)$ if I is a square free monomial ideal, but $(x_1\cdots x_d)^{p-1} \notin \mathfrak{m}^{[p]}$.

(2) Let $R = k[x,y,z]/(x^3+y^3+z^3)$. Then $(I^{[p]} : I) = ((x^3+y^3+z^3)^{p-1})$. If $p \equiv 1$ mod 3, then there is a term $(xyz)^{p-1}$ in the monomial expansion of $(x^3+y^3+z^3)^{p-1}$ with nonzero coefficient thus R is F-pure. On the other hand, if $p \equiv 2$ mod 3, then $(x^3+y^3+z^3)^{p-1} \in (x^p, y^p, z^p)$ so R is not F-pure.

We end this section with various exercises about F-pure rings.

Exercise 7. Show that R is F-pure if and only if R_P is F-pure for each $P \in \operatorname{Spec}(R)$.

Exercise 8. Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic $p > 0$. Show that R is F-pure if and only if $\widehat{R}$ is F-pure. What about the case that R is not necessarily F-finite?

Exercise 9. Let $S = \mathbb{Z}[x_1, \dots, x_d]$ and $p_1, \dots, p_\ell$ finitely many prime integers.

(1) Find $f \in S$ such that if p is a prime integer then $S/(f) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is F-pure if and only if $p \in \{p_1, \dots, p_\ell\}$.

(2) Find $g \in S$ such that if p is a prime integer then $S/(g) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is F-pure if and only if $p \notin \{p_1, \dots, p_\ell\}$.

4. Strongly F-regular rings

Definition 4.1. A reduced F-finite ring R is called *strongly F-regular* if for every $c \in R$ that is not in any minimal prime of R, there exists $e > 0$ such that the map $R \to R^{1/p^e}$ sending 1 to c^{1/p^e} splits as a map of R-modules.

We can define strongly F-regular rings beyond the F-finite case, see [HH94a] or [DS16]. Clearly, strongly F-regular rings are F-split.

Exercise 10. Show that a reduced F-finite ring R is strongly F-regular if and only if R_P is strongly F-regular for every $P \in \operatorname{Spec} R$. Moreover, an F-finite local ring $(R, \mathfrak{m})$ is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular.

Exercise 11. Let $(R, \mathfrak{m})$ be a strongly F-regular local ring. Show that R is a domain.

Theorem 4.2. *An F-finite regular ring is strongly F-regular.*

Proof. Both properties localize. So it is enough to show that if $(R, \mathfrak{m})$ is F-finite regular local, then R is strongly F-regular. By Kunz's theorem, R^{1/p^e} is a finite free R-module. For every $0 \neq c \in R$, there exists $e > 0$ such that $c^{1/p^e} \in R^{1/p^e}$ is part of a minimal basis of R^{1/p^e} over R: since otherwise $c^{1/p^e} \in \mathfrak{m}R^{1/p^e}$ for all e and thus $c \in \cap_e \mathfrak{m}^{[p^e]} = 0$ which is a contradiction. Since $c^{1/p^e} \in R^{1/p^e}$ is part of a minimal basis of R^{1/p^e} over R, the map $R \to R^{1/p^e}$ sending 1 to c^{1/p^e} splits. $\square$

Theorem 4.3. *If R is a direct summand of S and S is strongly F-regular (resp. F-pure), then R is strongly F-regular (resp. F-pure). In particular, direct summands of regular rings are strongly F-regular.*

Proof. We only prove the statement on strong F-regularity (the statement for F-purity is easier). For simplicity, we assume that S is a domain (we leave the general case as an exercise). Let $0 \neq c \in R$ be given. Since S is strongly F-regular we there exists $e > 0$ and an S-linear map $\phi\colon S^{1/p^e} \to S$ such that $\phi(c^{1/p^e}) = 1$. Let $\theta\colon S \to R$ be the splitting. Then $\theta \circ \phi\colon S^{1/p^e} \to R$ is an R-linear map sending c^{1/p^e} to 1. Restricting this map to R^{1/p^e} we get the desired splitting. $\square$

The above theorems allow us to write many examples of strongly F-regular rings:

Example 4.4.
(1) Let $R = k[x, y, z]/(xy - z^2)$. Then $R \cong k[s^2, st, t^2]$ is a direct summand of $S = k[s, t]$. Hence R is strongly F-regular. More generally, Veronese subrings of polynomial rings are strongly F-regular.
(2) Let $R = k[x, y, u, v]/(xy - uv)$. Then $R \cong k[a, b]\#k[c, d] \cong k[ac, ad, bc, bd]$ is a direct summand of $S = k[a, b, c, d]$. Hence R is strongly F-regular. More generally, Segre product of polynomial rings are strongly F-regular.

We also have some non-examples.

Example 4.5.
(1) Non-regular Stanley-Reisner rings are not strongly F-regular, since they are not domains.
(2) $R = k[[x, y, z]]/(x^2 + y^3 + z^7)$ is not strongly F-regular: use Fedder's criterion to check that R is not F-split.

Similar to Fedder's criterion for F-purity, we have an analog criterion for strong F-regularity. We leave the proof of this theorem as an exercise, the strategy is very similar to the proof of Fedder's criterion.

Theorem 4.6 ([Gla96])**.** *Let $(S, \mathfrak{m})$ be an F-finite regular local ring (resp. a standard graded polynomial ring over an F-finite field) and let $I \subseteq S$ be an ideal (resp. a homogeneous ideal). Then S/I is strongly F-regular if and only if for every c not in any minimal prime of I, there exists $e > 0$ such that $c(I^{[p^e]} : I) \nsubseteq \mathfrak{m}^{[p^e]}$.*

At this point one might ask that to check strong F-regularity, whether one needs to check the Frobenius splitting for every c not in any minimal prime of R. The next extremely useful result shows that it is enough to check this for certain c.

Theorem 4.7. *Let R be a reduced and F-finite ring. Suppose there exists c not in any minimal prime of R such that R_c is strongly F-regular (e.g., R_c is regular). Then R is strongly F-regular if and only if there exists $e > 0$ such that the map $R \to R^{1/p^e}$ sending 1 to c^{1/p^e} splits as R-modules.*

Proof. Given any $d \in R$ that is not in any minimal prime of R, the image of d is not in any minimal prime of R_c. Therefore, since R_c is strongly F-regular, there exists $e_0 > 0$ and a map $\phi \in \operatorname{Hom}_{R_c}(R_c^{1/p^{e_0}}, R_c)$ such that $\phi(d^{1/p^{e_0}}) = 1$. Since we have $\operatorname{Hom}_{R_c}(R_c^{1/p^{e_0}}, R_c) \cong \operatorname{Hom}_R(R^{1/p^{e_0}}, R)_c$, $\phi = \varphi/c^n$ for some $n > 0$ and some $\varphi \in \operatorname{Hom}_R(R^{1/p^{e_0}}, R)$. It follows that $\varphi(d^{1/p^{e_0}}) = c^n$. Now we pick $e_1 > 0$ such that $n/p^{e_1} < 1/p^e$, since $R \to R^{1/p^e}$ sending 1 to c^{1/p^e} splits, it follows that $R \to R^{1/p^{e_1}}$ sending 1 to $c^{n/p^{e_1}}$ splits (check this!). We call such splitting θ. Finally we consider the map $\theta \circ \varphi^{1/p^{e_1}}\colon R^{1/p^{e_0 e_1}} \to R^{1/p^{e_1}} \to R$, it sends $d^{1/p^{e_0 e_1}} \to c^{n/p^{e_1}} \to 1$. $\square$

Exercise 12. Let $R = k[x_1, \dots, x_d]/(x_1^n + \cdots + x_d^n)$. Show that R is strongly F-regular if $n < d$ and $p \gg 0$, and R is not strongly F-regular if $n \geq d$.

We next prove some nice properties of strongly F-regular rings.

Theorem 4.8. *Let R be a strongly F-regular ring. Then $R \to S$ splits for any module-finite extension S of R.*

Proof. We may assume that R is local and hence we may assume R is a domain. By killing a minimal prime of S, we may assume that S is also a domain. Now S is a torsion-free R-module, thus there exists an R-linear map $\theta\colon S \to R$ such that $\theta(1) = c \neq 0$. Since R is strongly F-regular, we can find e such that $R \to R^{1/p^e}$ sending 1 to c^{1/p^e} splits, call the splitting ϕ. Now consider the diagram with natural maps:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R^{1/p^e} \arrow[r] & S^{1/p^e} \end{tikzcd}$$

Now $\theta^{1/p^e}\colon S^{1/p^e} \to R^{1/p^e}$ sends 1 to c^{1/p^e}, thus $\phi \circ \theta^{1/p^e}$ sends $1 \in S^{1/p^e}$ to $1 \in R$. Therefore, $R \to S^{1/p^e}$ splits, this clearly implies $R \to S$ splits. $\square$

As an immediate consequence of the theorem, we obtain:

Corollary 4.9. *In characteristic $p > 0$, if R is regular, then $R \to S$ splits for any module-finite extension S of R.*

This result also holds in characteristic 0 (easy), and in mixed characteristic (difficult [And18]).

A very big open question in tight closure and F-singularity theory is that whether the converse of the above theorem is true.

Open Problem 2. *Let R be an F-finite domain. If $R \to S$ splits for any module-finite extension S of R, then is R strongly F-regular?*

This has an affirmative answer in the following cases:

(1) If R is Gorenstein by [HH94b].
(2) If R is $\mathbb{Q}$-Gorenstein by [Sin99a].
(3) If the anti-canonical cover of R is a Noetherian ring by [CEMS18].

For the readers not familiar with the terminology $\mathbb{Q}$-Gorenstein or anti-canonical cover, we just point out that there are implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) since every Gorenstein ring is $\mathbb{Q}$-Gorenstein and every $\mathbb{Q}$-Gorenstein ring has Noetherian anti-canonical cover.

Exercise 13. Show that if R is strongly F-regular, then R is normal. In particular, one-dimensional strongly F-regular rings are regular.

5. F-rational and F-injective rings

Let $I = (f_1, \ldots, f_n)$ be an ideal of R, then we have the Čech complex:

$$C^\bullet(f_1, \ldots, f_n; R) := 0 \to R \to \oplus_i R_{f_i} \to \cdots \to R_{f_1 f_2 \cdots f_n} \to 0.$$

The ith local cohomology module $H^i_I(R)$ is the ith cohomology of $C^\bullet(f_1, \ldots, f_n; R)$. It turns out that $H^i_I(R)$ only depends on the radical of I. Since the Frobenius endomorphism on R

naturally induces the Frobenius endomorphism on all localizations of R, it induces a natural Frobenius action on $C^\bullet(f_1,\dots,f_n;R)$, and hence it induces a natural Frobenius action on each $H^i_I(R)$.

We know from the definition that a ring homomorphism $R\to S$ induces a map $H^i_I(R)\to H^i_{IS}(S)$. If R is reduced, then it is easy to check that the natural Frobenius action on $H^i_I(R)$ (defined above) can be also realized as the composition

$$H^i_I(R)\to H^i_{IR^{1/p}}(R^{1/p})\cong H^i_I(R),$$

where the last isomorphism is induced by identifying $R^{1/p}$ with R.

We will be mostly interested in the case that $(R,\mathfrak{m})$ is local and $I=\mathfrak{m}$. In this case, we can compute $H^i_\mathfrak{m}(R)$ using the Čech complex on a system of parameters $x_1,\dots,x_d$ of R. For example, the top local cohomology module

$$H^d_\mathfrak{m}(R)=\frac{R_{x_1\cdots x_d}}{\sum_i \operatorname{Im}(R_{x_1\cdots\widehat{x_i}\cdots x_d})},$$

and the natural Frobenius action on $H^d_\mathfrak{m}(R)$ sends $\frac{r}{x_1^n\cdots x_d^n}$ to $\frac{r^p}{x_1^{np}\cdots x_d^{np}}$.

Definition 5.1. Let M be an R-module with a Frobenius action F (i.e., $F(rm)=r^pF(m)$ for all $r\in R$ and $m\in M$). An R-submodule $N\subseteq M$ is called *F-stable* if $F(N)\subseteq N$.

Exercise 14. Let M be an R-module with an Frobenius action and let $I\subseteq R$ be an ideal. Show that

(1) IM is F-stable.
(2) $H^0_I(M)$ is F-stable.

Exercise 15. Let $R=\mathbb{F}_p[[x,y]]$ and $W=\mathbb{F}_p\oplus H^2_\mathfrak{m}(R)$. Consider the Frobenius action F on W such that $F(1,0)=(1,x^{-p}y^{-1})$ and F is the natural Frobenius action $H^2_\mathfrak{m}(R)$. Show that

(1) F is injective on W/xW, and $x^{p-1}F$ is injective on W.
(2) There is no F-stable copy of $\mathbb{F}_p$ inside W.

Our definition of F-rational rings are not the original definition of Hochster–Huneke in [HH94b], but it is an equivalent definition for excellent local rings thanks to the work of Smith [Smi97]. In fact, Smith's characterization of F-rational rings turns out to be extremely powerful and has many geometric applications, see [Smi97] for more details.

Definition 5.2. An excellent local (or standard graded) ring $(R,\mathfrak{m})$ of dimension d is called *F-rational* if R is Cohen–Macaulay and the only F-stable submodules of $H^d_\mathfrak{m}(R)$ are 0 and $H^d_\mathfrak{m}(R)$, i.e., $H^d_\mathfrak{m}(R)$ is a simple object in the category of R-modules with a Frobenius action.

Example 5.3. Let $(R,\mathfrak{m}) = k[x_1,\ldots,x_d]$ or $k[[x_1,\ldots,x_d]]$. We have $H^d_{\mathfrak{m}}(R) \cong k[x_1^{-1},\ldots,x_d^{-1}]$. Suppose $0 \neq N \subseteq H^d_{\mathfrak{m}}(R)$ is F-stable, then $(x_1\cdots x_d)^{-1} \in N$ and thus $(x_1\cdots x_d)^{-p^e} \in N$ for all e. Hence $N = H^d_{\mathfrak{m}}(R)$.

An important result we prove now is that strongly F-regular rings are F-rational.

Theorem 5.4. *If an F-finite local ring $(R,\mathfrak{m})$ is strongly F-regular, then R is F-rational (and hence Cohen–Macaulay).*

Proof. First we show that R is Cohen–Macaulay. We proceed by induction on the dimension of R. Recall that every strongly F-regular local ring is a domain. If R is of dimension 0 then R is a field and hence strongly F-regular.

The properties of being F-finite, strongly F-regular, and Cohen–Macaulay are unaffected by completion. Thus we may further assume that R is complete and hence $R = S/I$ where S is a complete regular local ring. Moreover, the properties of being F-finite and strongly F-regular pass onto all localizations of R. Therefore by induction, we may assume R_P is Cohen–Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. By local duality, we have

$$H^i_{\mathfrak{m}}(R)^\vee \cong \operatorname{Ext}^{n-i}_S(R,S)$$

where $n = \dim S$. The module $\operatorname{Ext}^{n-i}_S(R,S)$ is Noetherian and its formation commutes with localization. We have

$$\operatorname{Ext}^{n-i}_S(R,S)_P \cong \operatorname{Ext}^{n-i}_{S_P}(S_P,R_P) = \operatorname{Ext}^{\dim S_P-(i-\dim R/P)}_{S_P}(S_P,R_P),$$

where we abuse notation and also use P to denote the pre-image of P in S. Now by local duality over S_P,

$$\operatorname{Ext}^{\dim S_P-(i-\dim R/P)}_{S_P}(S_P,R_P)^\vee \cong H^{i-\dim R/P}_{PR_P}(R_P).$$

Hence if $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $i < d = \dim R$, then $H^{i-\dim R/P}_{PR_P}(R_P) = 0$ since R_P is Cohen–Macaulay by induction (we are implicitly using that $\dim R/P + \operatorname{ht} P = d$, since R is a complete local domain) and thus $\operatorname{Ext}^{n-i}_S(R,S)_P = 0$. Therefore $\operatorname{Ext}^{n-i}_S(R,S)$ is supported only at the maximal ideal when $i < d$. By Matlis duality, $H^i_{\mathfrak{m}}(R)$ has finite length whenever $i < d$.

Let $0 \neq c \in \mathfrak{m}$ and let $i < d$. Since $H^i_{\mathfrak{m}}(R)$ has finite length, there exists N such that $c^N H^i_{\mathfrak{m}}(R) = 0$. Replacing c with c^N we may assume $cH^i_{\mathfrak{m}}(R) = 0$. Using the isomorphism $R \cong R^{1/p^e}$, we know that $c^{1/p^e} H^i_{\mathfrak{m}^{1/p^e}}(R^{1/p^e}) = c^{1/p^e} H^i_{\mathfrak{m}}(R^{1/p^e}) = 0$. Since R is strongly F-regular, there exists $e > 0$ and an R-linear map $R^{1/p^e} \to R$ such that the composition of the following maps is the identity map on R:

$$R \subseteq R^{1/p^e} \xrightarrow{\cdot c^{1/p^e}} R^{1/p^e} \to R.$$

Applying the i-th local cohomology functor $H^i_{\mathfrak{m}}(-)$ to the above composition of maps we see that the identity map on $H^i_{\mathfrak{m}}(R)$ factors through the zero map on $H^i_{\mathfrak{m}}(R^{1/p^e})$ and therefore $H^i_{\mathfrak{m}}(R) = 0$ whenever $i < d$. This proves that R is Cohen–Macaulay.

Finally we prove that R is F-rational. Let $N \neq H^d_{\mathfrak{m}}(R)$ be an F-stable submodule. Since R is a complete, Matlis duality implies we have $\omega_R \twoheadrightarrow N^\vee$. Since R is a domain, ω_R is torsion-free and thus $0 \neq \operatorname{Ann}_R N^\vee = \operatorname{Ann}_R N$. Pick $0 \neq c \in \operatorname{Ann}_R N$. Now we mimic the discussion above. Since R is strongly F-regular, there exists $e > 0$ and an R-linear map $R^{1/p^e} \to R$ such that the composition of the following maps is the identity map on R:

$$R \subseteq R^{1/p^e} \xrightarrow{\cdot c^{1/p^e}} R^{1/p^e} \to R.$$

Applying the d-th local cohomology functor $H^d_{\mathfrak{m}}(-)$ to the above composition of maps we obtain a commutative diagram:

$$\begin{tikzcd}
H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[dr, "F^e"] & H^d_{\mathfrak{m}}(R^{1/p^e}) \arrow[r, "\cdot c^{1/p^e}"] & H^d_{\mathfrak{m}}(R^{1/p^e}) \\
 & H^d_{\mathfrak{m}}(R) \arrow[u, "\cong"] \arrow[r, "\cdot c"] & H^d_{\mathfrak{m}}(R) \arrow[u, "\cong"]
\end{tikzcd}$$

such that the first row is injective. However, since N is F-stable and $cN = 0$, the composition from top left to bottom right is the zero map. This shows that $N = 0$ and thus R is F-rational. $\square$

As an immediate consequence of the theorem, we obtain:

Corollary 5.5. *In characteristic $p > 0$, direct summands of regular rings are Cohen–Macaulay.*

This result also holds in characteristic 0 [HH95], and in mixed characteristic [HM18] (based on [And18]).

Definition 5.6. A local (or standard graded) ring $(R, \mathfrak{m})$ is called *F-injective* if the natural Frobenius action on $H^i_{\mathfrak{m}}(R)$ is injective for all i.

Exercise 16. Show that if $(R, \mathfrak{m})$ is F-rational or F-pure, then R is F-injective.

Both F-rational and F-injective singularities localize, though this is not obvious to prove. We give an explanation when R is F-finite. We write $R = S/I$ for a regular local ring. By F-finiteness and duality, one can check that the Frobenius action F on $H^i_{\mathfrak{m}}(R)$ corresponds to a Cartier morphism C on $\operatorname{Ext}^{n-i}_S(R, S)$ (i.e., $C(r^p x) = rC(x)$ for all $x \in \operatorname{Ext}^{n-i}_S(R, S)$). One next shows that F is injective if and only if C is surjective, and that $H^d_{\mathfrak{m}}(R)$ is simple in the category of R-modules with a Frobenius action if and only if $\operatorname{Ext}_S^{n-d}(R,S)$ is simple in the category of Cartier modules (i.e., R-modules with a Cartier morphism). Now both F-rationality and F-injectivity can be characterized using the Cartier structure on $\operatorname{Ext}_S^{n-i}(R,S)$, which is finitely generated and behaves well under localization.

Exercise 17. Let $(R,\mathfrak{m})$ be an F-finite local ring. Prove the following:

(1) If R is F-rational, then R is normal.
(2) If R is F-injective, then R is reduced.

Exercise 18. Let $(R,\mathfrak{m})$ be a local (resp. standard graded) ring. Prove that if R is F-rational, then so is $R[[x]]$ (resp. $R[x]$). Prove the analogous result for strong F-regularity, F-purity, and F-injectivity.

One can ask that, similar to strong F-regularity and F-purity, whether a direct summand of F-rational or F-injective ring remains F-rational or F-injective. Unfortunately, Watanabe [Wat97] constructed an example of a direct summand of an F-rational ring that is not F-injective. This leaves an open question:

Open Problem 3. *Are direct summands of F-rational rings Cohen–Macaulay?*

Exercise 19. Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a module-finite extension that is split. Show that if S is Cohen–Macaulay, then R is also Cohen–Macaulay.

Exercise 20. One cannot expect that direct summands of Cohen–Macaulay rings are Cohen–Macaulay in general. Let R be the Segre product $(k[x,y,z]/(x^3+y^3+z^3))\#k[s,t]$, which is the subring of the hypersurface $S = k[x,y,z,s,t]/(x^3+y^3+z^3)$ generated by xs, ys, zs, xt, yt, zt. Then R is a direct summand of S. Show that R is not Cohen–Macaulay, and show that S is not F-rational.

We end this section with a quick summary of the relations between the F-singularities we have introduced so far:

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{normal} \\
 & F\text{-pure} \arrow[r, Rightarrow] & F\text{-injective} \arrow[r, Rightarrow] & \text{reduced}
\end{tikzcd}$$

It is known that for Gorenstein rings, F-injective rings are F-pure and F-rational rings are strongly F-regular [HH94b]. However, Watanabe [Wat91] constructed examples of F-rational rings that are not F-pure, and examples of F-rational and F-pure rings that are not strongly F-regular. See also [HH94b].

6. The deformation problem

A central and interesting question in the study of singularities is how they behave under deformation: if $\operatorname{Spec} R$ is the total space of a fibration over a curve, then the special fiber of this fibration is a variety with coordinate ring R/xR for a nonzerodivisor x of R. The question is whether the singularity type of the total space $\operatorname{Spec} R$ is no worse than the singularity type as the special fiber $\operatorname{Spec} R/xR$.

This deformation question has been studied in details for F-singularities: it is proved that F-rationality always deforms [HH94a], and that both F-pure and strongly F-regular singularities fail to deform in general [Fed83], [Sin99b], [Sin99c].

Theorem 6.1. *Let $(R,\mathfrak{m})$ be a local ring and x a nonzerodivisor on R. Then*

(1) If R/xR is Cohen–Macaulay and F-injective, then R is Cohen–Macaulay and F-injective.

(2) If R/xR is F-rational, then R is F-rational.

Proof. We first prove (1). It is clear that R is Cohen–Macaulay. It is enough to show that the natural Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective. The commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"'] & R \arrow[r] \arrow[d, "F^e"'] & R/xR \arrow[r] \arrow[d, "F^e"'] & 0 \\
0 \arrow[r] & R \arrow[r, "\cdot x"] & R \arrow[r] & R/xR \arrow[r] & 0
\end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/xR) \arrow[r] \arrow[d, "F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/xR) \arrow[r] & H^d_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^d_{\mathfrak{m}}(R) \arrow[r] & 0
\end{tikzcd}$$

If the middle map is not injective, then we pick $\eta \in \operatorname{Soc}(H^i_{\mathfrak{m}}(R)) \cap \operatorname{Ker}(x^{p^e-1}F^e)$ and it is easy to see that η comes from $H^{d-1}_{\mathfrak{m}}(R/xR)$. But this contradicts the injectivity of F^e on $H^{d-1}_{\mathfrak{m}}(R/xR)$. Thus $x^{p^e-1}F^e$ and hence F^e is injective on $H^d_{\mathfrak{m}}(R)$.

We next prove (2). Since F-rational rings are Cohen–Macaulay and F-injective. We know from the proof of (1) that $x^{p^e-1}F^e$ is injective on $H^d_{\mathfrak{m}}(R)$. Let N be an F-stable submodule of $H^d_{\mathfrak{m}}(R)$. Consider $N \supseteq xN \supseteq x^2N \supseteq \cdots$. This chain stabilizes since $H^d_{\mathfrak{m}}(R)$ is Artinian. Set $L = \cap_n x^n N = x^t N$ for all $t \gg 0$.

If $L = 0$, then $x^{p^e-1}F^e(N) \subseteq x^{p^e-1}N = L = 0$ for $e \gg 0$. Hence $N = 0$ by the injectivity of $x^{p^e-1}F^e$. If $L \neq 0$, then $0 \neq L \cap H^{d-1}_{\mathfrak{m}}(R/xR)$ is an F-stable submodule of $H^{d-1}_{\mathfrak{m}}(R/xR)$.

Since R/xR is F-rational, $H^{d-1}_{\mathfrak{m}}(R/xR) \subseteq L$ (viewed as submodules of $H^d_{\mathfrak{m}}(R)$). I claim that this implies $H^d_{\mathfrak{m}}(R)/L \xrightarrow{\cdot x} H^d_{\mathfrak{m}}(R)/L$ is injective (hence we must have $H^d_{\mathfrak{m}}(R) = L = N$). To see this, suppose $x\eta \in L$ for some $\eta \in H^d_{\mathfrak{m}}(R)$, then $x\eta = x\eta'$ for some $\eta' \in L$ since $L = xL$. But then $x(\eta - \eta') = 0$ and thus $\eta - \eta' \in H^{d-1}_{\mathfrak{m}}(R/xR) \subseteq L$, so $\eta \in L$. □

Whether F-injectivity deforms in general is still not known:

Open Problem 4. *Let $(R, \mathfrak{m})$ be a local ring and x a nonzerodivisor on R. If R/xR is F-injective, then is R also F-injective?*

To this date, the best partial result towards the above question is obtained in [HMS14], where it is shown that F-purity deforms to F-injectivity (note that F-purity itself does not deform in general, we will give an explicit example at the end of this section).

Theorem 6.2 ([HMS14])**.** *Let $(R, \mathfrak{m})$ be a local ring and x a nonzerodivisor on R. If R/xR is F-pure, then R is F-injective.*

We give a proof of this theorem using the following:

Theorem 6.3 ([Ma14])**.** *If $(R, \mathfrak{m})$ is F-pure, then for all i and all F-stable submodules $N \subseteq H^i_{\mathfrak{m}}(R)$, the natural Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is injective.*

Outline of proof. There are 4 steps.

(1) By passing to the completion, we may assume R is F-split.
(2) Show that if $R \to S$ is split, y is an element of $H^i_{\mathfrak{m}}(R)$, and N is a submodule of $H^i_{\mathfrak{m}}(R)$, then $y \in N$ provided that the image of y in $H^i_{\mathfrak{m}S}(S)$ is contained in the S-span of the image of N in $H^i_{\mathfrak{m}S}(S)$.
(3) Show that the natural Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is injective for all F-stable N if and only if for all $y \in H^i_{\mathfrak{m}}(R)$, y is in the R-span of $\langle F(y), F^2(y), F^3(y), \ldots\ldots \rangle$.
(4) Now for all $y \in H^i_{\mathfrak{m}}(R)$, let N_e be the R-span of $\langle F^e(y), F^{e+1}(y), F^{e+2}(y), \ldots\ldots \rangle$. We know the descending chain $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ stabilizes since $H^i_{\mathfrak{m}}(R)$ is Artinian. So $F^e(y) \in N_{e+1}$ for all $e \gg 0$. We apply step (2) to the eth Frobenius map F^e: $R \to R$ and note that the span of the image of N_1 is precisely N_{e+1}. Hence we have $y \in N_1$ and we are done by step (3). □

Proof of Theorem 6.2. The strategy is similar to the Cohen–Macaulay case. The commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"'] & R \arrow[r] \arrow[d, "F^e"'] & R/xR \arrow[r] \arrow[d, "F^e"'] & 0 \\ 0 \arrow[r] & R \arrow[r, "\cdot x"] & R \arrow[r] & R/xR \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & \cdots \\
0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & \cdots
\end{tikzcd}$$

Note that $\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R))$ is an F-stable submodule of $H^{i-1}_{\mathfrak{m}}(R/xR)$. So by the above theorem, F^e is injective on $H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R))$. Now by the same argument as in Theorem 6.1, this implies that $x^{p^e-1}F^e$ and hence F^e is injective on $H^i_{\mathfrak{m}}(R)$. □

Exercise 21. Let $R = k[[x, y, z, w]]/(xy, xz, y(z - w^2))$. Prove the following:

(1) R is Cohen–Macaulay and w is a nonzerodivisor on R.
(2) R/wR is F-pure.
(3) R is F-injective, but not F-pure.

7. Determinantal rings

The goal of this section is to explain in detail that generic determinantal rings of maximal minors are F-rational. In fact, generic determinantal rings over F-finite fields (of arbitrary minors) are strongly F-regular [HH94b], we will also comment on how to extend the method to obtain this stronger result.

The following criterion of F-rationality is well-known as Watanabe's criterion, it is a very powerful tool to check F-rationality in the graded case. The analogous criterion for rational singularities in characteristic 0 was first proved by Watanabe [Wat83].

Theorem 7.1 (Watanabe's criterion)**.** *Let $(R, \mathfrak{m})$ be a standard graded k-algebra. Then R is F-rational if and only if*

(1) R is Cohen–Macaulay.
(2) R_P is F-rational for all homogeneous prime $P \neq \mathfrak{m}$.
(3) $a(R) := \max\{n | H^d_{\mathfrak{m}}(R)_n \neq 0\} < 0$.
(4) R is F-injective.

Outline of proof. First suppose R is F-rational, then (1) and (4) clearly hold. (2) holds since F-rationality localizes as we explained. (3) holds because $H^d_{\mathfrak{m}}(R)_{\geq 0}$ is an F-stable submodule of $H^d_{\mathfrak{m}}(R)$ and it cannot be the whole $H^d_{\mathfrak{m}}(R)$ (check this!).

Now suppose R satisfies (1)–(4). To prove R is F-rational, the crucial step here is to use (2) and graded local duality (and the relation between Frobenius structure on $H^d_{\mathfrak{m}}(R)$ and Cartier structure on ω_R) to show that any graded F-stable submodule of $H^d_{\mathfrak{m}}(R)$ has finite length. Now by (4) any such graded F-stable submodule must concentrate in degree 0, but then it vanishes by (3). $\square$

Proposition 7.2. *Let $S = k[x_{ij}|1 \le i \le m, 1 \le j \le n]$ be a polynomial ring in $m \times n$ variables with $m \le n$. Let I_m be the ideal of S generated by $m \times m$ minors of the matrix $[x_{ij}]_{1\le i\le m, 1\le j\le n}$. Then $R = S/I_m$ is F-rational.*

Proof. We apply Watanabe's criterion. We first prove (2). For any homogeneous prime $P \neq \mathfrak{m}$, there exists $x_{ij} \notin P$. Without loss of generality we may assume $x_{11} \notin P$. After inverting the element x_{11}, we may perform row and column operations to transform our matrix:

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & 0 & \dots & 0 \\ 0 & x'_{22} & \dots & x'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x'_{m2} & \dots & x'_{mn} \end{bmatrix}$$

where $x'_{ij} = x_{ij} - \frac{x_{i1}x_{1j}}{x_{11}}$. The ideal $I_m S_{x_{11}}$ is generated by $(m-1)\times(m-1)$ minors of the second displayed matrix. Therefore, $R_{x_{11}} = S_{x_{11}}/I_m S_{x_{11}} \cong (S'/I'_{m-1})[x_{11}, \frac{1}{x_{11}}]$ where $S' = k[x_{ij}|2 \le i \le m, 2 \le j \le n]$ and I'_{m-1} denotes the ideal generated by the $(m-1)\times(m-1)$ minors of the matrix $[x'_{ij}]$. By induction on m, we know that S'/I'_{m-1} is F-rational, thus so is $(S'/I'_{m-1})[x_{11}]$. Now R_P can be viewed as a localization of $R_{x_{11}} \cong (S'/I'_{m-1})[x_{11}, \frac{1}{x_{11}}]$, therefore it is also the localization of $(S'/I'_{m-1})[x_{11}]$, so R_P is F-rational.

Now we prove (1), (3) and (4). We comment that (1) and (3) are well-known. However we will also reprove them along the way we prove (4). We need to use the following result from combinatorial commutative algebra:

Theorem 7.3 ([Stu90]). *The maximal minors of $[x_{ij}]_{1\le i\le m, 1\le j\le n}$ form a Gröbner basis of I_m with respect to the term order $x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots\cdots > x_{mn}$.*

At this point we follow the standard construction as in [Eis95, 15.16 and 15.17]. We choose an appropriate weight function λ such that $\mathrm{in}_\lambda(I_m) = \mathrm{in}_>(I_m)$. Let $\widetilde{I}$ be the λ-homogenization of I_m in $S[t]$. We have

$$(S[t]/\widetilde{I}) \otimes_{k[t]} k(t) \cong R \otimes_k k(t) \text{ and } (S[t]/\widetilde{I})/t \cong S/\mathrm{in}_>(I_m).$$

Therefore if we can show that $S/\mathrm{in}_>(I_m)$ is Cohen–Macaulay and F-injective, then so is $S[t]/\widetilde{I}$ by the graded version of Theorem 6.1. But then $R \otimes_k k(t)$ is Cohen–Macaulay and F-injective and hence R is Cohen–Macaulay and F-injective (check this!). Since the maximal minors form a Gröbner basis,

$$\mathrm{in}_>(I_m) = (x_{1i_i}x_{2i_2}\cdots x_{mi_m}|1 \le i_1 < i_2 < \cdots i_m \le n)$$

is a square-free monomial ideal. Thus $S/\operatorname{in}_>(I_m)$ is F-pure and hence F-injective. To see $S/\operatorname{in}_>(I_m)$ is Cohen-Macauay, note that we can write $\operatorname{in}_>(I_m) = J_1 \cap J_2$ where

$$J_1 = (x_{11}) + (x_{1i_i}x_{2i_2}\cdots x_{mi_m} | 2 \le i_1 < i_2 < \cdots i_m \le n),$$

$$J_2 = (x_{2i_2}\cdots x_{mi_m} | 2 \le i_2 < \cdots i_m \le n).$$

For example, when $m = 3$ and $n = 4$, we have

$$(x_{11}x_{22}x_{33}, x_{11}x_{22}x_{34}, x_{11}x_{23}x_{34}, x_{12}x_{23}x_{34}) = (x_{11}, x_{12}x_{23}x_{34}) \cap (x_{22}x_{33}, x_{22}x_{34}, x_{23}x_{34}).$$

By induction on the size of the matrix, we see that S/J_1 and S/J_2 are both Cohen–Macaulay (since the monomials that occur in J_1 and J_2 correspond to the initial ideal of the minors in a smaller size matrix!). Observe that $J_1 + J_2 = (x_{11}) + J_2$ so $S/(J_1 + J_2)$ is also Cohen–Macaulay. Moreover, one can check that $\operatorname{ht} J_1 = \operatorname{ht} J_2 = \operatorname{ht}(J_1 + J_2) - 1$. Now $S/\operatorname{in}_>(I_m)$ is Cohen–Macaulay follows from the following elementary exercise:

Exercise 22. Let A be a local (or standard graded) ring and let J_1 and J_2 be two (homogeneous) ideals in A. If A/J_1, A/J_2, and $A/(J_1 + J_2)$ are all Cohen–Macaulay with $\dim R/J_1 = \dim R/J_2 = 1 + \dim R/(J_1 + J_2)$, then $A/(J_1 \cap J_2)$ is Cohen–Macaulay.

We have completed the proof of (1) and (4). It remains to prove (3). I claim that the following sequence form a system of parameters on R.

$$(\dagger): \quad x_{21}, x_{31}, x_{32}, \ldots, x_{m1}, x_{m2}, \ldots, x_{m,m-1}$$

$$x_{1,n-m+2}, x_{1,n-m+3}, \ldots, x_{1n}, x_{2,n-m+3}, \ldots, x_{2n}, \ldots, x_{m-1,n}$$

$$x_{11} - x_{22}, x_{11} - x_{33}, \ldots, x_{11} - x_{mm}, x_{12} - x_{23}, \ldots, x_{1,n-m+1} - x_{m,n}$$

Note that the above sequence has length

$$m(m-1) + (m-1)(n-m+1) = (m-1)(n+1) = \dim R,$$

and killing the $(\dagger)$ corresponds to the following specialization of the matrix:

$$\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ x_{31} & x_{32} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \ldots & x_{mn} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1,n-m+1} & 0 & \ldots & \ldots & 0 \\ 0 & x_{11} & x_{12} & \ldots & x_{1,n-m+1} & 0 & \ldots & 0 \\ 0 & 0 & x_{11} & \ldots & x_{1,n-m} & x_{1,n-m+1} & \ldots, & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & x_{1,n-m+1} \end{bmatrix}.$$

For example, when $m = 3$ and $n = 4$, $(\dagger)$ becomes:

$$x_{21}, x_{31}, x_{32}, x_{13}, x_{14}, x_{24}, x_{11} - x_{22}, x_{11} - x_{33}, x_{12} - x_{23}, x_{12} - x_{34}$$

which corresponds to the following specialization of the matrix:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & x_{12} & 0 & 0 \\ 0 & x_{11} & x_{12} & 0 \\ 0 & 0 & x_{11} & x_{12} \end{bmatrix}.$$

It is easy to check that the ideal generated by the maximal minors of the right hand side matrix is $(x_{11}, x_{12}, \ldots, x_{1,n-m+1})^m$. Thus after we kill the sequence (†) we have a graded Artinian ring $\overline{R} := k[x_{11}, x_{12}, \ldots, x_{1,n-m+1}]/(x_{11}, x_{12}, \ldots, x_{1,n-m+1})^m$ (and obviously $a(\overline{R}) = m-1$). Hence the (†) is a system of parameters of R. Since we already proved that R is Cohen–Macaulay, (†) is a regular sequence on R.

Exercise 23. Let $(R, \mathfrak{m})$ be a standard graded Cohen–Macaulay k-algebra. Let x be a homogenous nonzerodivisor of degree t. Prove that $a(R/xR) = a(R) + t$.

By repeatedly applying the above exercise, we have:

$$a(R) = a(\overline{R}) - \dim R = a(\overline{R}) - (m-1)(n+1) = (m-1) - (m-1)(n+1) = -n(m-1) < 0.$$

This completes the proof. $\square$

We can use the same strategy to prove that S/I_t is F-rational where I_t denotes the ideal generated by $t \times t$ minors of the matrix $[x_{ij}]_{1 \le i \le m, 1 \le j \le n}$: the argument for (2) is exactly the same; the $t \times t$ minors still form a Gröbner basis with respect to the given term order [Stu90] so the initial ideal is still a square-free monomial ideal and hence F-injective; to show the initial ideal is Cohen–Macaulay, we can use Hochster's criterion [Hoc72] that a Stanley-Reisner ring is Cohen–Macaulay if the corresponding simplicial complex is shellable; and the a-invariant can also be computed using the combinatorial structure [BH92]. Finally, to see that S/I_t is strongly F-regular, we note that we can enlarge the $m \times n$ matrix to an $n \times n$ matrix and consider the corresponding quotients S'/I_t of $t \times t$ minors in the $n \times n$ matrix. Then $S/I_t \to S'/I_t$ splits (since we can map the new variables to zero), thus S/I_t is strongly F-regular provided S'/I_t is strongly F-regular by Theorem 4.3. But S'/I_t is Gorenstein and thus F-rationality of S'/I_t implies the strong F-regularity of S'/I_t [HH94b]. We encourage the interested reader to carry out the details.

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